INTEGER TORSION IN LOCAL COHOMOLOGY, AND QUESTIONS ON TIGHT CLOSURE THEORY

by

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ABSTRACT

Grothendieck's theory of local cohomology has applications to basic questions such as determining the minimal number of polynomial equations needed to define an algebraic set. These modules are typically not finitely generated, and a question of Huneke asks whether they have finitely many associated prime ideals. This was settled in the negative by Singh, who constructed a local cohomology module that has prime-torsion for each prime integer. We extend this work in Chapter 1 by showing that the module in question contains a copy of each finitely generated abelian group. Moreover, the module has a natural fine grading, and we are able to show that each finitely generated abelian group embeds into a single graded component.

In Chapter 2 we study F-injectivity, i.e., the property that the Frobenius action on local cohomology modules is injective. We obtain an effective criterion to determine if a diagonal subalgebra of a bigraded hypersurface is F-injective; such subalgebras provide a rich source of examples of various F-singularities, and thus form a natural testing ground for various questions and conjectures related to classes of singularities defined via the Frobenius map.

Chapter 3 is an investigation of the tight closure properties of rings of invariants of finite groups acting linearly on polynomial rings over fields of positive characteristic.

Dedicated to my family, especially my mother, and gwa

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CHAPTER 1

TORSION IN LOCAL COHOMOLOGY MODULES

Local cohomology modules are, in general, not finitely generated over the ambient ring. Towards a better understanding of the structure of these modules, Huneke [Hu] asked if they have finitely many associated prime ideals. Singh [Si2] showed that the answer is negative by constructing a local cohomology module that has a p-torsion element for each prime integer p, and hence has infinitely many associated prime ideals. This existence of p-torsion is equivalent to the statement that $\mathbb{Z}/p\mathbb{Z}$ embeds into the local cohomology module for each prime integer p. We show in this chapter that, in fact, each finitely generated abelian group embeds into the local cohomology module in question.

1.1 Introduction

Let $R = K[x_1, \ldots, x_n]$ be the ring of polynomials with coefficients in an algebraically closed field K. Given polynomials $f_1, f_2, \ldots, f_m$ in R, the zero set of these polynomials is the subset of K^n given by

$$V(f_1, \ldots, f_m) = \{\boldsymbol{p} \in K^n \mid f_i(\boldsymbol{p}) = 0 \text{ for each } 1 \leqslant i \leqslant n\}.$$

Zero sets of polynomials include familiar geometrical objects, for example

$$V(x^2 + y^2 - 1)$$

is a circle. It is readily seen that $V(f_1, \ldots, f_m)$ agrees with $V(I)$, where I is the ideal of R generated by the polynomials $f_1, \ldots, f_m$. Moreover, a polynomial f vanishes at a point of K^n precisely is some positive integer power f^k vanishes at that point. It follows that

$$V(I) = V(\sqrt{I}),$$

where $\sqrt{I}$ denotes the *radical* of I, i.e., the ideal

$$\{f \in R \mid f^k \in I \text{ for some } k \geqslant 1\}.$$

Hilbert's Nullstellensatz states that when working over an algebraically closed field, one has an equality $V(I) = V(J)$ if and only if $\sqrt{I} = \sqrt{J}$. It is natural to ask what is the least number of generators needed to define an algebraic set; the Nullstellensatz reformulates this as the least number of elements that generate an ideal up to radical.

Example 1.1. Let $R = K[w,x,y,z]$, and I be the ideal $(w,x) \cap (y,z) = (wy,wz,xy,xz)$. Then $V(I)$ is the union of two planes in K^4 that intersect at the origin. One may easily verify that

$$\sqrt{(wy,xz,wz+xy)} = I\,,$$

and it follows that

$$V(wy,wz,xy,xz) = V(wy,xz,wz+xy)\,.$$

Thus, this zero set of four polynomials is also the zero set of three polynomials. It is not the zero set of two polynomials—see Example 1.17—where this is verified using local cohomology.

Huneke [Hu, Problem 4] asked whether local cohomology modules of noetherian rings have finitely many associated prime ideals. The answer to this is negative: Singh [Si2] constructed an example where, for R a hypersurface, $H^3_I(R)$ has p-torsion elements for each prime integer p, and hence has infinitely many associated primes; see Theorem 1.22. For equicharacteristic as well as local rings R, the first examples of local cohomology modules $H^j_I(R)$ with infinitely many associated primes were constructed by Katzman [Ka]. He showed that for

$$R = \frac{K[s,t,u,v,x,y]}{\big((sux-tvy)(ux-vy)\big)}\,,$$

the local cohomology module $H^2_{(x,y)}(R)$ has infinitely many associated primes. The rings in [Ka] are not integral domains; subsequently, Singh and Swanson [SS] constructed families of graded hypersurfaces over arbitrary fields, for which a local cohomology module has infinitely many associated primes. For example they show that for

$$R = \frac{K[r,s,t,u,v,w,x,y,z]}{\big(su^2x^2+sv^2y^2+tuxvy+rw^2z^2\big)}\,,$$

the module $H^3_{(x,y,z)}(R)$ has infinitely many associated primes; the hypersurface R is a unique factorization domain that has rational singularities if K has characteristic zero, and is F-regular if K has positive characteristic.

On the other hand, if R is a regular ring, then there are affirmative answers to Huneke's question in several cases and, in fact, Lyubeznik has conjectured that for regular rings R, each local cohomology module $H^j_I(R)$ has finitely many associated primes, see [Ly1, Remark 3.7]. Huneke and Sharp [HS] have proved this for regular rings R that contain a field of prime characteristic. If R is a regular affine or local ring containing a field of characteristic zero, Lyubeznik [Ly1] showed that each $H^j_I(R)$ has only finitely many associated prime ideals. The case of unramified regular local rings of mixed characteristic was settled by Lyubeznik in [Ly2]. However, Lyubeznik's conjecture remains unresolved for polynomial rings over rings of integers, where p-torsion is a central issue.

1.2 Local cohomology

In this section we discuss Koszul, Čech, and local cohomology modules. We begin with the Koszul complex.

Definition 1.2. Let $x \in R$. The *Koszul complex* on x, denoted by $K^\bullet(x, R)$, is the complex

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow 0\,.$$

The Koszul complex on a sequence of elements $\boldsymbol{x} = x_1, \dots, x_n$ of R is defined to be

$$K^\bullet(\boldsymbol{x}, R) = K^\bullet(x_1, R) \otimes_R \cdots \otimes_R K^\bullet(x_n, R)\,.$$

Example 1.3. Let x and y be elements of R. The Koszul complexes $K^\bullet(x, R)$ and $K^\bullet(y, R)$ are respectively

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow 0$$

and

$$0 \longrightarrow R \xrightarrow{y} R \longrightarrow 0\,.$$

From this, we calculate $K^\bullet(x, y, R)$ to be the complex

$$0 \longrightarrow R \longrightarrow R^2 \longrightarrow R \longrightarrow 0\,,$$

where the map $R \longrightarrow R^2$ is given by

$$\begin{pmatrix} -y \\ x \end{pmatrix},$$

and the map $R^2 \longrightarrow R$ by the matrix

$$\begin{pmatrix} x & y \end{pmatrix}.$$

One can easily verify that the composition of these two maps is the zero map. We now define the Koszul complex on an R-module M.

Definition 1.4. Let $\boldsymbol{x} = x_1, \ldots, x_n$ be elements of R, and M an R-module. The Koszul complex of $\boldsymbol{x}$ on M is the complex

$$K^\bullet(\boldsymbol{x}, M) = K^\bullet(\boldsymbol{x}, R) \otimes_R M \,.$$

We define the *Koszul cohomology* of $\boldsymbol{x}$ on M as

$$H^j(\boldsymbol{x}, M) = H^j(K^\bullet(\boldsymbol{x}, M)) \,.$$

One can see that

$$H^0(x, M) = \{m \in M \mid mx = 0\} \,,$$

and hence that x is a nonzerodivisor on M if and only if $H^0(x, M) = 0$. An element $x \in R$ is said to be *M-regular* if $xm \neq 0$ for all nonzero $m \in M$, and $xM \neq M$. Thus, an element x in R is M-regular if and only if $H^0(x, M)$ is zero, and $H^1(x, M) = M/xM$ is nonzero.

The Koszul cohomology modules $H^0(\boldsymbol{x}, M)$ and $H^n(\boldsymbol{x}, M)$ can be expressed in terms of familiar objects:

$$H^0(\boldsymbol{x}, M) = \big(0 :_M (x_1, \ldots, x_n)\big) \quad \text{and} \quad H^n(\boldsymbol{x}, M) = M/(x_1, \ldots, x_n)M \,.$$

It follows that the ideal $(x_1, \ldots, x_n)$ annihilates $H^0(x_1, \ldots, x_n, M)$ and $H^n(x_1, \ldots, x_n, M)$. More generally, one has:

Theorem 1.5. *Let $x_1, \ldots, x_n$ be elements of R, and let M be an R-module. For each integer j, the ideal $(x_1, \ldots, x_n)$ annihilates $H^j(x_1, \ldots, x_n, M)$.*

Definition 1.6. A sequence of elements $x_1, \ldots, x_n$ of R is said to be *M-regular* if

1. x_1 is M-regular, and x_i is $(M/\sum_{k=1}^{i-1} x_k M)$-regular, and
2. $M/(\sum_{k=1}^{n} x_k M) \neq 0$.

If we drop the second condition, the sequence is *weakly M-regular*.

Example 1.7. The order of the elements in a regular sequence matters: let $R = K[x, y, z]$ be a polynomial ring over a field K. Then the sequence $x - 1, xy, xz$ is a regular sequence, but $xy, xz, x - 1$ is not a regular sequence: the element xz is a zerodivisor on $R/(xy)R$, as it is killed by y.

Theorem 1.8. *Let M be a nonzero R-module, and $I = (x_1, \ldots, x_n)$ an ideal of R.*

1. *If I contains a weakly M-regular sequence of length t, then $H^j(x_1, \ldots, x_n, M) = 0$ for each $j < t$.*

2. If the R-module M is finitely generated and $IM \neq M$, then

$$\operatorname{depth}_R(I, M) = \min\{j \mid H^j(x_1, \dots, x_n, M) \neq 0\}.$$

Proof. We prove part 1. Choose $y \in I$ so that y is a nonzerodivisor on M. This yields the exact sequence of R-modules

$$0 \longrightarrow M \xrightarrow{y} M \longrightarrow M/yM \longrightarrow 0.$$

The Koszul complex $K^\bullet(\boldsymbol{x}, R)$ consists of projective modules; tensoring with the above exact sequence gives us the exact sequence of complexes

$$0 \longrightarrow K^\bullet(x_1, \dots, x_n, M) \xrightarrow{y} K^\bullet(x_1, \dots, x_n, M) \longrightarrow K^\bullet(x_1, \dots, x_n, M/yM) \longrightarrow 0.$$

This induces a cohomological long exact sequence

$$\begin{aligned} \cdots \longrightarrow H^j(x_1, \dots, x_n, M) &\xrightarrow{y} H^j(x_1, \dots, x_n, M) \longrightarrow H^j(x_1, \dots, x_n, M/yM) \\ \longrightarrow H^{j+1}(x_1, \dots, x_n, M) &\longrightarrow \cdots. \end{aligned}$$

By Theorem 1.5, the element y annihilates each $H^j(x_1, \dots, x_n, M)$ giving us short exact sequences

$$0 \longrightarrow H^j(x_1, \dots, x_n, M) \longrightarrow H^j(x_1, \dots, x_n, M/yM) \longrightarrow H^{j+1}(x_1, \dots, x_n, M) \longrightarrow 0.$$

We proceed by induction on t. Let $y_1, \dots, y_t$ be a weakly M-regular sequence in I. It follows that $y_2, \dots, y_t$ is weakly M/y_1M-regular. We apply the induction hypothesis to M/y_1M and get $H^j(x_1, \dots, x_n, M/y_1M) = 0$ for $j < t-1$. The exact sequence displayed above with $y = y_1$ gives the desired result. $\square$

For a finitely generated module M over a local ring, it follows that if $x_1, \dots, x_n$ is an M-regular sequence, then so is any permutation.

We now discuss the Čech complex which, like the Koszul complex, is built from tensoring smaller complexes.

Let x be an element of R; we use R_x or $R[1/x]$ to denote the ring of fractions where x is inverted. There is a canonical map

$$i \colon R \longrightarrow R_x \quad \text{given by} \quad r \longmapsto r/1.$$

Definition 1.9. Let x be an element of the ring R. The Čech complex $\check{C}^\bullet(x,R)$ is the complex

$$0 \longrightarrow R \xrightarrow{i} R_x \longrightarrow 0\,.$$

Here, the module R is in cohomological degree 0, and R_x in cohomological degree 1. For a sequence of elements $\boldsymbol{x} = x_1, \dots, x_n$ of R, the Čech complex is given by

$$\check{C}^\bullet(\boldsymbol{x},R) = \check{C}^\bullet(x_1,R) \otimes_R \cdots \otimes_R \check{C}^\bullet(x_n,R)\,.$$

For an R-module M we define

$$\check{C}^\bullet(\boldsymbol{x},M) = \check{C}^\bullet(\boldsymbol{x},R) \otimes_R M\,.$$

The j-th Čech cohomology of $\boldsymbol{x}$ on M is defined to be

$$\check{H}^j(\boldsymbol{x},M) = H^j(\check{C}^\bullet(\boldsymbol{x},R))\,.$$

Example 1.10. Let x, y be elements of a ring R. Then $\check{C}^\bullet(x,R)$ and $\check{C}^\bullet(y,R)$ are respectively the complexes:

$$0 \longrightarrow R \xrightarrow{i_x} R_x \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow R \xrightarrow{i_y} R_y \longrightarrow 0\,.$$

Taking the tensor product of these gives the complex $\check{C}^\bullet(x,y,R)$, i.e.,

$$0 \longrightarrow R \longrightarrow R_x \oplus R_y \longrightarrow R_{xy} \longrightarrow 0\,.$$

We now discuss local cohomology, and summarize some facts that may be found in [ILL].

Definition 1.11. Let R be a noetherian ring and I an ideal of R. For an R-module M, set

$$\Gamma_I(M) = \{m \in M \mid I^t m = 0 \ \text{ for some } \ t \geqslant 0\}\,.$$

Each R-module homomorphism $\varphi\colon M \longrightarrow M'$ induces

$$\Gamma_I(\varphi)\colon \Gamma_I(M) \longrightarrow \Gamma_I(M')\,.$$

Local cohomology $H^j_I(-)$ is the j-th right derived functor of $\Gamma_I(-)$.

Example 1.12. We remarked earlier that local cohomology modules are, in general, not finitely generated over the ambient ring. For example if $R = \mathbb{C}[x]$ then

$$H^1_{(x)}(\mathbb{C}[x]) = R_x/R = \bigoplus_{i>0} \mathbb{C}x^{-i}\,,$$

which is not a finitely generated R-module.

Theorem 1.13. *Let M be an R-module.*

1. *One has $H^0_I(M) = \Gamma_I(M)$, and $H^j_I(M)$ is I-torsion for each j.*

2. *If $\sqrt{I} = \sqrt{J}$, then $H^j_I(M) = H^j_J(M)$ for each j.*

3. *Let $\{M_\lambda\}$ be R-modules. For each integer j, one has $H^j_I(\oplus_\lambda M_\lambda) = \oplus_\lambda H^j_I(M_\lambda)$.*

4. *An exact sequence of R-modules $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ induces an exact sequence of local cohomology modules*

$$\cdots \longrightarrow H^j_I(M') \longrightarrow H^j_I(M) \longrightarrow H^j_I(M'') \longrightarrow H^{j+1}_I(M') \longrightarrow \cdots .$$

Theorem 1.14. *For M an R-module, there is a natural isomorphism*

$$H^j_I(M) \cong \varinjlim_{k \in \mathbb{N}} \operatorname{Ext}^j_R\left(R/I^k, M\right),$$

where the maps in the direct limit system are those induced by the canonical surjections $R/I^{k+1} \longrightarrow R/I^k$.

Theorem 1.15. *Let I be generated by elements $x_1, \ldots, x_n$. For each R-module M and each integer j, we have isomorphisms*

$$H^j_I(M) \cong \check{H}^j(x_1, \ldots, x_n, M).$$

In particular, $H^j_I(R)$ is isomorphic to the j-th cohomology module of the Čech complex

$$0 \longrightarrow R \longrightarrow \bigoplus_{i=1}^{n} R_{x_i} \longrightarrow \bigoplus_{i<j} R_{x_i x_j} \longrightarrow \cdots \longrightarrow R_{x_1 \cdots x_n} \longrightarrow 0.$$

Corollary 1.16. *Let R be a ring, and $I = (x_1, \ldots, x_n)$ an ideal of R. For each R-module M, one has $H^j_I(M) = 0$ for $j > n$, and*

$$H^n_I(M) = M_{x_1 \cdots x_n} / \sum_{i=1}^{n} \operatorname{Im}(M_{x_1 \cdots \widehat{x_i} \cdots x_n}).$$

Example 1.17. We return to Example 1.1 and show that the ideal $I = (wy, wz, xy, xz)$ in $R = K[w, x, y, z]$ is not the radical of a 2-generated ideal. In view of Theorem 1.13 (2) and the above corollary, it suffices to show that $H^3_I(R)$ is nonzero. This is a straightforward calculation using the Mayer-Vietoris sequence

$$H^3_{(w,x)}(R) \oplus H_{(y,z)}(R) \longrightarrow H^3_I(R) \longrightarrow H^4_{(w,x,y,z)}(R),$$

since $H^3_{(w,x)}(R) = 0 = H_{(y,z)}(R)$ and $H^4_{(w,x,y,z)}(R) \neq 0$.

Example 1.18. If $S = \mathbb{C}[u, v, w, x, y, z]$, the algebraic set

$$V(vz - wy, wx - uz, uy - vx)$$

is the set of 2×3 complex matrices of rank less than 2. To see that this is not the zero set of two polynomials, it suffices to check that $H^3_J(S)$ is nonzero where J is the ideal of S generated by the matrix minors $\Delta_1 = vz - wy$, $\Delta_2 = wx - uz$, and $\Delta_3 = uy - vx$.

The group $G = SL_2(\mathbb{C})$ acts linearly on S with ring of invariants

$$S^G = \mathbb{C}[\Delta_1, \Delta_2, \Delta_3]\,,$$

which is a polynomial ring of dimension three, since the minors are algebraically independent over $\mathbb{C}$. Hence $H^3_{(\Delta_1,\Delta_2,\Delta_3)}(S^G)$ is nonzero. Since G is linearly reductive, S^G is a direct summand of S as an S^G-module, and it follows that $H^3_{(\Delta_1,\Delta_2,\Delta_3)}(S^G)$ is a nonzero direct summand of $H^3_J(S)$.

Lemma 1.19. *Let R be a ring, and $I = (x_1, \dots, x_n)$ an ideal of R. The element*

$$\left[\frac{f}{x_1^m \cdots x_n^m}\right]$$

of the local cohomology module $H^n_I(R)$ is zero if and only if there exists a non-negative integer k such that

$$f(x_1 \cdots x_n)^k \in (x_1^{m+k}, \dots, x_n^{m+k})R\,.$$

Proof. If there exist elements $r_i \in R$ with

$$f(x_1 \cdots x_n)^k = \sum_{i=1}^{n} r_i x_i^{m+k}\,,$$

then

$$\left[\frac{f}{(x_1 \cdots x_n)^m}\right] = \left[\frac{f(x_1 \cdots x_n)^k}{(x_1 \cdots x_n)^{m+k}}\right] = \sum_{i=1}^{n} \left[\frac{r_i x_i^{m+k}}{x_1^{m+k} \cdots x_m^{m+k}}\right].$$

But then this element is zero in $H^n_I(R)$, since it comes from $\check{C}^{n-1}(\boldsymbol{x}, R)$.

Conversely, suppose $\left[\frac{f}{(x_1 \cdots x_n)^m}\right] = 0$ in $H^n_I(R)$. Then the element

$$\frac{f}{(x_1 \cdots x_n)^m} \in R_{x_1 \cdots x_n}$$

lies in the image of $\check{C}^{n-1}(\boldsymbol{x}, R)$, so

$$\frac{f}{(x_1 \cdots x_n)^m} = \sum_{i=1}^{n} \frac{r_i}{(x_1 \cdots \widehat{x}_i \cdots x_n)^{m+k}}$$

for some $r_i \in R$ and $k \geqslant 0$. It follows that

$$f(x_1 \cdots x_n)^k = \sum_{i=1}^{n} r_i x_i^{m+k} \quad \text{in } R_{x_1 \cdots x_n}\,,$$

and hence that

$$f(x_1 \cdots x_n)^{k+l} \in (x_1^{m+k+l}, \ldots, x_n^{m+k+l})R$$

for some $l \geqslant 0$. □

We record the following determinant computation from [Mu, page 682] that we will use later.

Lemma 1.20. *For nonnegative integers n, a, k, one has*

$$\det \begin{vmatrix} \binom{n}{a+k} & \binom{n}{a+k+1} & \cdots & \binom{n}{a+2k} \\ \binom{n}{a+k-1} & \binom{n}{a+k} & \cdots & \binom{n}{a+2k-1} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \binom{n}{a} & \binom{n}{a+1} & \cdots & \binom{n}{a+k} \end{vmatrix} = \frac{\binom{n}{a+k}\binom{n+1}{a+k}\cdots\binom{n+k}{a+k}}{\binom{a+k}{a+k}\binom{a+k+1}{a+k}\cdots\binom{a+2k}{a+k}}\,.$$

Lastly, we recall the definition of associated prime ideals:

Definition 1.21. A prime ideal P of R is *associated* to M if it is the annihilator of an element of M. We write $\operatorname{Ass}_R(M)$ for the set of associated primes of M.

A prime ideal P is the annihilator of an element m of M precisely if P is the kernel of the homomorphism

$$R \longrightarrow M \qquad \text{where } r \longmapsto rm\,.$$

Thus,

$$\operatorname{Ass}_R(M) = \{P \in \operatorname{Spec} R \mid R/P \hookrightarrow M\}\,.$$

1.3 Main results

The example below constructed by Singh shows that there are infinitely many associated primes of a local cohomology module. His result is equivalent to the statement that $\mathbb{Z}/p\mathbb{Z}$ embeds into a cohomology module for each prime integer p. In this setting we generalize his result.

Theorem 1.22 ([Si2]). *Consider the hypersurface*

$$R = \frac{\mathbb{Z}[u,x,v,y,w,z]}{(ux+vy+wz)}$$

and the ideal $I = (x,y,z)R$. Then, for each prime integer p, the element

$$\lambda_p = \left[\frac{(ux)^p + (vy)^p + (wz)^p}{p(xyz)^p}\right]$$

of the local cohomology module $H^3_I(R)$ is p-torsion; consequently $H^3_I(R)$ has infinitely many associated prime ideals.

Throughout this section, R will denote the above hypersurface. This hypersurface has a $\mathbb{Z}^4$-grading where

$$\begin{aligned} \deg x &= (1,0,0,0), & \deg u &= (-1,0,0,1),\\ \deg y &= (0,1,0,0), & \deg v &= (0,-1,0,1),\\ \deg z &= (0,0,1,0), & \deg w &= (0,0,-1,1). \end{aligned}$$

For nonnegative integers a,b,c,d, we shall prove that the graded component

$$\left[H^3_I(R)\right]_{(-a,-b,-c,d)}$$

of the local cohomology module is isomorphic, as an abelian group, to

$$\left[\frac{\mathbb{Z}[A,B]}{(A^a, B^b, (A+B)^c)}\right]_d,$$

where $\mathbb{Z}[A,B]$ is $\mathbb{N}$-graded with $\deg A = 1 = \deg B$.

We also show that for each prime integer p, the element

$$\lambda_p = \left[\frac{(ux)^p + (vy)^p + (wz)^p}{p(xyz)^p}\right]$$

of $H^3_I(R)$ is not p-divisible.

Lemma 1.23. *The graded component*

$$\left[H^3_I(R)\right]_{(-a,-b,-c,d)}$$

of $H^3_I(R)$ vanishes unless a,b,c,d are all nonnegative; if they are all nonnegative, then it is generated, as an abelian group, by the elements

$$\left[\frac{(ux)^i(vy)^{d-i}}{x^a y^b z^c}\right]$$

where $0 \leqslant i \leqslant d$.

In particular, each $\mathbb{Z}^4$-graded component of $H^3_I(R)$ is a finitely generated abelian group.

Proof. Any element in the local cohomology module of degree $(-a,-b,-c,d)$ is a $\mathbb{Z}$-linear combination of elements of the form

$$\left[\frac{u^i v^j w^k}{x^{a-i} y^{b-j} z^{c-k}}\right]$$

where i,j,k are nonnegative and $d=i+j+k$. Note that

$$\left[\frac{u^i v^j w^k}{x^{a-i} y^{b-j} z^{c-k}}\right] = \left[\frac{(ux)^i (vy)^j (wz)^k}{x^a y^b z^c}\right].$$

Since $wz=-ux-vy$ in R, we have

$$\left[\frac{(ux)^i (vy)^j (wz)^k}{x^a y^b z^c}\right] = (-1)^k \sum_{t=0}^{k} \binom{k}{t} \left[\frac{(ux)^{i+t} (vy)^{j+k-t}}{x^a y^b z^c}\right].$$

Since $j+k=d-i$, the assertions follow. $\square$

Remark 1.24. We contrast Lemma 1.23 with an example from [SW2].

Set $S=\mathbb{Z}[u,v,w,x,y,z]$ and $J=(vz-wy, wx-uz, uy-vx)$. Give the ideal J a homogeneous with respect to the $\mathbb{N}^4$-grading where

$$\begin{aligned} \deg x &= (1,0,0,0), & \deg u &= (1,0,0,1),\\ \deg y &= (0,1,0,0), & \deg v &= (0,1,0,1),\\ \deg z &= (0,0,1,0), & \deg w &= (0,0,1,1). \end{aligned}$$

It follows that $H^3_J(S)$ is $\mathbb{Z}^4$-graded. It is proved in [SW2] that each nonzero $\mathbb{Z}^4$-graded component of $H^3_J(S)$ is a $\mathbb{Q}$-vector space, and hence not a finitely generated abelian group.

Remark 1.25. Let S be a polynomial ring in finitely many variables over $\mathbb{Z}$. It remains an open question whether a local cohomology module $H^j_J(S)$ can have p-torsion for infinitely many prime integers p. However, given finitely many primes $p_1,\dots,p_k$, [SW1, Example 5.11] shows that there exists a polynomial ring $S=\mathbb{Z}[x_1,\dots,x_n]$, with a monomial ideal J, such that $H^{n-2}_J(S)$ has prime torsion precisely for the prime integers $p_1,\dots,p_k$. The module $H^{n-2}_J(S)$ has a $\mathbb{Z}^n$-grading, since J is a monomial ideal.

We now return to the study of the module $H^3_I(R)$ as in Theorem 1.22.

Theorem 1.26. *For nonnegative integers a,b,c,d, the graded component*

$$\left[H^3_I(R)\right]_{(-a,-b,-c,d)}$$

of the local cohomology module $H^3_I(R)$ is isomorphic, as an abelian group, to

$$\left[\frac{\mathbb{Z}[A,B]}{(A^a, B^b, (A+B)^c)}\right]_d.$$

Proof. This is surjective by Lemma 1.23. Note that $\mathbb{Z}[A,B]_d$ is a free abelian group on the generators A^iB^{d-i} for $0 \leqslant i \leqslant d$. Consider the group homomorphism

$$\begin{aligned} \theta\colon \mathbb{Z}[A,B]_d &\longrightarrow \left[H^3_I(R)\right]_{(-a,-b-c,d)} \quad \textit{where} \\ A^iB^{d-i} &\longmapsto \left[\frac{(ux)^i(vy)^{d-i}}{x^ay^bz^c}\right] . \end{aligned}$$

Suppose $\sum_{i=0}^d r_iA^iB^{d-i}$ belongs to $\ker\theta$, where $r_i \in \mathbb{Z}$. Then

$$\sum_{i=0}^d r_i\left[\frac{(ux)^i(vy)^{d-i}}{x^ay^bz^c}\right] = 0\,,$$

which, by Lemma 1.19 implies that

$$\sum_{i=0}^d r_i(ux)^i(vy)^{d-i}(xyz)^N \in (x^{a+N}, y^{b+N}, z^{c+N})R$$

for some integer $N \geqslant 0$. Thus, there exist homogeneous elements f, g, h in R with

$$\sum_{i=0}^d r_i(ux)^i(vy)^{d-i}(xyz)^N = fx^{a+N} + gy^{b+N} + hz^{c+N}\,.$$

Since the left hand side of this equation has degree (N,N,N,d), it follows that f, g, and h are, respectively, multiples of $u^ay^Nz^N$, $v^bx^Nz^N$, and $w^cx^Nz^N$. Hence

$$\sum_{i=0}^d r_i(ux)^i(vy)^{d-i}(xyz)^N = (xyz)^N\big(F(ux)^a + G(vy)^b + H(wz)^c\big)\,,$$

where F, G, and H, have degrees $(0,0,0,d-a)$, $(0,0,0,d-b)$, and $(0,0,0,d-c)$ respectively. Since R is a domain, it follows that

$$\sum_{i=0}^d r_i(ux)^i(vy)^{d-i} = \big(F(ux)^a + G(vy)^b + H(wz)^c\big)\,.$$

Note that the elements F, G, H, ux, vy, wz belong to the subring

$$\bigoplus_{k\geqslant 0}[R]_{(0,0,0,k)}$$

of R, which is an $\mathbb{N}$-graded ring isomorphic to

$$\mathbb{Z}[A,B,C]/(A+B+C)\,,$$

the isomorphism being given by $ux \longmapsto A$, $vy \longmapsto B$, and $wz \longmapsto C$. Thus,

$$\sum_{i=0}^d r_iA^iB^{d-i} \in \big(A^a, B^b, (A+B)^c\big)\,,$$

which shows that θ induces an injective map

$$\left[\frac{\mathbb{Z}[A,B]}{\big(A^a, B^b, (A+B)^c\big)}\right]_d \longrightarrow \left[H^3_I(R)\right]_{(-a,-b,-c,d)}\,.$$

$\square$

We are now ready to construct arbitrary n-torsion elements:

Corollary 1.27. *For each positive integer n, the element*

$$\left[\frac{u^{n-1}v}{xyz^n}\right]$$

of the local cohomology module $H^3_I(R)$ is n-torsion.

Proof. By Theorem 1.26, it suffices to verify that the image of $A^{n-1}B$ is n-torsion in

$$\left[\frac{\mathbb{Z}[A,B]}{(A^n, B^2, (A+B)^n)}\right]_n .$$

The binomial expansion

$$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}$$

shows that

$$\left[\frac{\mathbb{Z}[A,B]}{(A^n, B^2, (A+B)^n)}\right]_n = \left[\frac{\mathbb{Z}[A,B]}{(A^n, B^2, nA^{n-1}B)}\right]_n ,$$

and the assertion follows. □

Theorem 1.28. *For each prime integer p, the graded component*

$$\left[H^3_I(R)\right]_{(-p,-p,-p,p)}$$

of the local cohomology module $H^3_I(R)$ is isomorphic, as an abelian group, to

$$\mathbb{Z}^{p-2} \oplus \mathbb{Z}/p\mathbb{Z} .$$

Proof. Theorem 1.26 implies that

$$\left[H^3_I(R)\right]_{(-p,-p,-p,p)} \cong \left[\frac{\mathbb{Z}[A,B]}{(A^p, B^p, (A+B)^p)}\right]_p .$$

The group on the right hand side has $p-1$ generators, namely

$$AB^{p-1},\ A^2B^{p-2},\ \ldots,\ A^{p-1}B ,$$

and one relation, namely

$$\sum_{i=1}^{p-1} \binom{p}{i} A^i B^{p-i} .$$

Hence

$$\left[H^3_I(R)\right]_{(-p,-p,-p,p)} \cong \frac{\mathbb{Z}^{p-1}}{\left(\binom{p}{1}, \binom{p}{2}, \ldots, \binom{p}{p-1}\right)} .$$

Since the greatest common divisor of the binomial coefficients $\binom{p}{i}$ with $1 \leqslant i \leqslant p-1$ is p, it follows that

$$\left[H^3_I(R)\right]_{(-p,-p,-p,p)} \cong \mathbb{Z}^{p-2} \oplus \mathbb{Z}/p\mathbb{Z}\,.$$

□

Remark 1.29. The element

$$\lambda_p = \left[\frac{(ux)^p + (vy)^p + (wz)^p}{p(xyz)^p}\right]$$

considered by Singh in Theorem 1.22 has degree $(-p,-p,-p,p)$. By the previous theorem, it follows that the p-torsion component of

$$\left[H^3_I(R)\right]_{(-p,-p,-p,p)}$$

is spanned by λ_p. It also follows that λ_p is a p-torsion element that is not p-divisible.

We next use Theorem 1.26 to determine the group structure of the graded components of $H^3_I(R)$, and provide presentation matrices. Since the computation is symmetric in a, b, c, we assume that these are ordered as $a \leqslant b \leqslant c$.

Theorem 1.30. *Assume that $a \leqslant b \leqslant c$.*

1. *If $a+b \leqslant d+1$, then $\left[H^3_I(R)\right]_{(-a,-b,-c,d)}$ is zero.*
2. *If $a+b \geqslant d+2$ and $c > d$, then $\left[H^3_I(R)\right]_{(-a,-b,-c,d)}$ is a torsion-free abelian group, and its rank is given by*

$$\operatorname{rank}\left[H^3_I(R)\right]_{(-a,-b,-c,d)} = \begin{cases} d+1 & \text{if } d < a\,, \\ a & \text{if } a \leqslant d < b\,, \\ a+b-d-1 & \text{if } a \leqslant b \leqslant d\,. \end{cases}$$

3. *If $a+b \geqslant d+2$ and $c \leqslant d$, then $\left[H^3_I(R)\right]_{(-a,-b,-c,d)}$ has presentation matrix*

$$\begin{pmatrix} \binom{c}{c-b+1} & \binom{c}{c-b+2} & \cdots & \binom{c}{c+a-d-1} \\ \binom{c}{c-b+2} & \binom{c}{c-b+3} & \cdots & \binom{c}{c+a-d} \\ \vdots & \vdots & \vdots & \vdots \\ \binom{c}{d-b+1} & \binom{c}{d-b+2} & \cdots & \binom{c}{a-1} \end{pmatrix}.$$

Proof. (1) If $a+b \leqslant d+1$, then

$$\left[\frac{\mathbb{Z}[A,B]}{(A^a,B^b)}\right]_d = 0\,,$$

and hence

$$\left[H^3_I(R)\right]_{(-a,-b,-c,d)} = \left[\frac{\mathbb{Z}[A,B]}{(A^a,B^b,(A+B)^c)}\right]_d = 0\,.$$

(2) Since $c > d$, the graded component in question is isomorphic to

$$\left[\frac{\mathbb{Z}[A,B]}{(A^a,B^b,(A+B)^c)}\right]_d = \left[\frac{\mathbb{Z}[A,B]}{(A^a,B^b)}\right]_d\,,$$

A count of the nonzero monomials gives the desired result.

(3) The generators

$$A^{d-b+1}B^{b-1},\ A^{d-b+2}B^{b-2},\ \ldots,\ A^{a-1}B^{d-a+1}$$

correspond to the $a+b-d-1$ columns of the presentation matrix, and the rows correspond to the $d-c+1$ relations

$$\begin{gathered}(A+B)^c A^{d-c}\\ (A+B)^c A^{d-c-1}B\\ \vdots\\ (A+B)^c B^{d-c}\,.\end{gathered}$$

Taking binomial expansions of the above elements, one arrives at the entries of the presentation matrix. $\square$

Example 1.31. The presentation matrix for $\left[H^3_I(R)\right]_{(-4,-4-4,5)}$ is

$$\begin{pmatrix}\binom{4}{1} & \binom{4}{2}\\ \binom{4}{2} & \binom{4}{3}\end{pmatrix} = \begin{pmatrix}4 & 6\\ 6 & 4\end{pmatrix},$$

which is equivalent, after elementary row and column operations, to

$$\begin{pmatrix}2 & 0\\ 0 & 10\end{pmatrix}.$$

It follows that

$$\left[H^3_I(R)\right]_{(-4,-4-4,5)} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z}\,.$$

As generators for the 2-torsion subgroup $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, we may choose

$$2\left[\frac{(ux)^2(vy)^3}{(xyz)^4}\right] + 3\left[\frac{(ux)^3(vy)^2}{(xyz)^4}\right] \qquad \text{and} \qquad 3\left[\frac{(ux)^2(vy)^3}{(xyz)^4}\right] + 2\left[\frac{(ux)^3(vy)^2}{(xyz)^4}\right]\,.$$

Remark 1.32. Suppose $1 \leqslant a \leqslant b \leqslant c \leqslant d$ and $a+b+c = 2d+2$. Then $\left[H^3_I(R)\right]_{(-a,-b,-c,d)}$ is presented by the square matrix:

$$M = \begin{pmatrix} \binom{c}{c-b+1} & \binom{c}{c-b+2} & \cdots & \binom{c}{d-b+1} \\ \binom{c}{c-b+2} & \binom{c}{c-b+3} & \cdots & \binom{c}{d-b+2} \\ \vdots & \vdots & \vdots & \vdots \\ \binom{c}{d-b+1} & \binom{c}{d-b+2} & \cdots & \binom{c}{a-1} \end{pmatrix}.$$

The determinant of M is nonzero by Lemma 1.20. Consider the exact sequence

$$\mathbb{Z}^{d-c+1} \longrightarrow \mathbb{Z}^{d-c+1} \longrightarrow \left[H^3_I(R)\right]_{(-a,-b,-c,d)} \longrightarrow 0\,,$$

where the map $\mathbb{Z}^{d-c+1} \longrightarrow \mathbb{Z}^{d-c+1}$ is given by M. We tensor with $\mathbb{Q}$ to get

$$\mathbb{Q}^{d-c+1} \longrightarrow \mathbb{Q}^{d-c+1} \longrightarrow \left[H^3_I(R)\right]_{(-a,-b,-c,d)} \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow 0\,.$$

Since the matrix M is invertible over $\mathbb{Q}$, we see that

$$\left[H^3_I(R)\right]_{(-a,-b,-c,d)} \otimes_{\mathbb{Z}} \mathbb{Q} = 0\,,$$

i.e., that every element of $\left[H^3_I(R)\right]_{(-a,-b,-c,d)}$ is a torsion element.

1.4 Embedding groups into $H^3_I(R)$

We conclude this chapter by showing that every finitely generated abelian group embeds into a single graded component of the local cohomology module $H^3_I(R)$; the hypersurface R and ideal I are as in Theorem 1.22. We prove this by studying the presentation matrix Theorem 1.30. In this setting the entries of the matrix will be integers. We will need th following lemma which we now prove.

Let A and B be $m \times n$ matrices with integer entries. We say A and B are *equivalent* if there exist matrices $M \in \mathrm{GL}_m(\mathbb{Z})$ and $N \in \mathrm{GL}_n(\mathbb{Z})$ with $B = MAN$. Note that A and B are equivalent precisely if they are presentation matrices for isomorphic abelian groups.

Lemma 1.33. *Every $m \times n$ matrix A with integer entries is equivalent to one where all the nonzero entries occur on the main diagonal.*

Fix a nonzero integer q. If each entry of A is a multiple of q, then A is equivalent to a matrix where all the nonzero entries occur on the main diagonal, and are multiples of q.

Proof. We perform elementary row and column operations on A that consist of permuting rows (or columns) and adding a multiple of a row (or column) to another; an elementary row operation may be performed by left-multiplication by an invertible matrix, and an elementary column operation by right-multiplication by an invertible matrix. Thus, these preserve the equivalence class of a matrix.

Let $A = (a_{ij})$. Using the Euclidean algorithm over $\mathbb{Z}$, perform elementary row and column operations on A until one obtains a nonzero entry a_{11} with $|a_{11}|$ as small as possible. It then follows that a_{11} divides each entry in the first row and in the first column. Thus, one may perform further elementary row and column operations to obtain an equivalent matrix of the form

$$\begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}.$$

We now repeat this process with the lower right $m-1 \times n-1$ matrix.

If each entry of the original matrix is a multiple of q, this is preserved by all elementary row and column operations, and hence the resulting matrix has multiples of q along the main diagonal, and zero entries elsewhere. $\square$

Theorem 1.34. *Each finitely generated abelian group embeds into a graded component of the local cohomology module $H^3_I(R)$.*

Proof. Given a finitely generated abelian group

$$\mathbb{Z}^t \oplus \frac{\mathbb{Z}}{q_1\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{q_n\mathbb{Z}},$$

set $q = \prod_{i=1}^n q_i$. The group above is a subgroup of

$$\mathbb{Z}^t \oplus \left(\frac{\mathbb{Z}}{q\mathbb{Z}}\right)^n,$$

so it suffices to determine a graded component $\left[H^3_I(R)\right]_{(-a,-b,-c,d)}$ with an embedding

$$\mathbb{Z}^t \oplus \left(\frac{\mathbb{Z}}{q\mathbb{Z}}\right)^n \hookrightarrow \left[H^3_I(R)\right]_{(-a,-b,-c,d)}$$

If $n = 0$, take $a = b = c = t$, and $d = t-1$. Theorem 1.30 implies that $\left[H^3_I(R)\right]_{(-a,-b,-c,d)}$ is a free abelian group of rank t.

Now suppose $n \geqslant 1$. Set

$$a = 2n + t\,, \qquad b = c = (2n + t - 1)!\,q\,, \qquad \text{and} \qquad d = n + c - 1\,.$$

By Theorem 1.30, the presentation matrix for $\left[H_I^3(R)\right]_{(-a,-b,-c,d)}$ is

$$\begin{pmatrix} \binom{c}{1} & \binom{c}{2} & \cdots & \binom{c}{n+t} \\ \binom{c}{2} & \binom{c}{3} & \cdots & \binom{c}{n+t+1} \\ \vdots & \vdots & \vdots & \vdots \\ \binom{c}{n} & \binom{c}{n+1} & \cdots & \binom{c}{2n+t-1} \end{pmatrix}.$$

We claim that the entries in the presentation matrix are all integer multiples of q. This follows since these entries are

$$\binom{c}{k} = \binom{c-1}{k-1}\frac{c}{k} = q\binom{c-1}{k-1}\frac{(2n+t-1)!}{k},$$

where $1 \leqslant k \leqslant 2n+t-1$.

The matrix has n rows and $n+t$ columns, and the first n columns are linearly independent by Lemma 1.20; thus, the matrix has rank n. By Lemma 1.33, the presentation matrix is equivalent to a matrix of the form

$$\begin{pmatrix} qd_1 & 0 & 0 & \cdots & 0 & 0 & \dots & 0 \\ 0 & qd_2 & 0 & \cdots & 0 & 0 & \dots & 0 \\ 0 & 0 & qd_3 & \cdots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & qd_n & 0 & \dots & 0 \end{pmatrix}.$$

Since it has rank n, it follows that each d_i is nonzero. Thus,

$$\left[H_I^3(R)\right]_{(-a,-b,-c,d)} \cong \mathbb{Z}^t \oplus \frac{\mathbb{Z}}{qd_1\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{qd_n\mathbb{Z}},$$

which contains an isomorphic copy of

$$\mathbb{Z}^t \oplus \left(\frac{\mathbb{Z}}{q\mathbb{Z}}\right)^n.$$

We have thus shown that every finitely generated abelian group embeds into a single graded component of the local cohomology module $H_I^3(R)$. $\square$

CHAPTER 2

F-INJECTIVITY OF MULTIGRADED RINGS

In this chapter we study the F-injectivity of diagonal subalgebras of bigraded rings: such rings arise as homogeneous coordinate rings of blow-ups of projective varieties, and also as rings of invariants of natural toric actions. We obtain an effective criterion to determine the F-injectivity of a diagonal subalgebra of a bigraded hypersurface; these subalgebras provide a rich source of examples of various F-singularities, [KSSW], and thus form a natural testing ground for various questions and conjectures related to classes of singularities defined via the Frobenius map.

2.1 Introduction

Let X be a projective variety over a field K, with homogeneous coordinate ring A. Let $\mathfrak{a} \subset A$ be a homogeneous ideal, and $V \subset X$ the closed subvariety defined by $\mathfrak{a}$. Using $\mathfrak{a}_g$ to denote the K-vector space of homogeneous elements of $\mathfrak{a}$ of degree g, the rings $K[(\mathfrak{a}^h)_g]$ are homogeneous coordinate rings for the blow-up of X along V, when $g \gg h > 0$. The connection with diagonal subalgebras stems from the fact that if $\mathfrak{a}^h$ is generated by elements of degree less than or equal to g, then

$$K[(\mathfrak{a}^h)_g] \cong \bigoplus_{k \geqslant 0} A[\mathfrak{a}t]_{(gk,hk)} .$$

The papers [GG, GGH, GGP] use diagonal subalgebras in studying blow-ups of projective space at finite sets of points. Various ring-theoretic properties of diagonal subalgebras are studied by Simis, Trung, and Valla in [STV], and by Conca, Herzog, Trung, and Valla in [CHTV]. The Cohen-Macaulay property of diagonal subalgebras is investigated also in [HHR, Lv1, Lv2], and [LvZ].

Hochster and Huneke developed tight closure theory in the paper [HH2]. The definition of tight closure is:

Definition 2.1. Let I be an ideal of a ring R of characteristic $p > 0$, and let R° denote the complement of the minimal primes of R. We say that $x \in I^*$, the *tight closure* of I, if there exists a $c \in R^\circ$ such that $cx^q \in I^{[q]}$ for all $q = p^e$. If $I^* = I$, then we say that I is tightly closed.

They defined F-regular and F-rational rings in terms of tight closure: a ring R is *weakly F-regular* if every ideal of R is tightly closed, and is *F-regular* if every localization is weakly F-regular. A ring R is *F-rational* if every parameter ideal of R is tightly closed.

For the theory of tight closure, we refer to the papers [HH1, HH2, HH3] and [HH4]. We summarize a few results about F-rational and F-regular rings:

Theorem 2.2. *The following hold for rings containing a field of prime characteristic:*

1. *Regular rings are F-regular.*

2. *Direct summands of F-regular rings are F-regular.*

3. *F-rational rings are normal; an F-rational ring that is a homomorphic image of a Cohen-Macaulay ring is Cohen-Macaulay.*

4. *F-rational Gorenstein rings are F-regular.*

5. *Let R be an $\mathbb{N}$-graded ring that is finitely generated over a field R_0. If R is weakly F-regular, then it is F-regular.*

Proof. For (1) and (2) see [HH2, Theorem 4.6] and [HH2, Proposition 4.12] respectively; (3) is part of [HH3, Theorem 4.2], and for (4) see [HH3, Corollary 4.7], Lastly, (5) is [LS, Corollary 4.4]. □

There is a closure operation on ideals that is typically smaller than tight closure, and is called Frobenius closure:

Definition 2.3. Let R be a ring of prime characteristic p. We say that $x \in I^F$, the *Frobenius closure* of I, if there exists $q = p^e$ such that $x^q \in I^{[q]}$.

A ring R is *F-pure* if the Frobenius endomorphism $F\colon R \longrightarrow R$ is pure; this definition is due to Hochster and Roberts [HR].

The study of F-injectivity dates back to Fedder and Watanabe, [FW]:

Definition 2.4. A local or graded ring $(R, \mathfrak{m})$ of prime characteristic is said to be *F-injective* if the Frobenius endomorphism of R induces an injective homomorphism of local cohomology modules

$$F\colon H^i_m(R) \longrightarrow H^i_m(R) \qquad \text{for each } 0 \leqslant i \leqslant \dim(R)\,.$$

If R is F-pure, it follows that it is F-injective.

Via reduction modulo p, these notions are closely related to the singularities in characteristic zero that occur in the minimal model program. The characteristic zero aspects of tight closure are developed in [HH5]. Let K be a field of characteristic zero. A finitely generated K-algebra $R = K[x_1, \ldots, x_m]/\mathfrak{a}$ is of *weakly F-regular type* if there exists a finitely generated $\mathbb{Z}$-algebra $A \subseteq K$, and a finitely generated free A-algebra

$$R_A = A[x_1, \ldots, x_m]/\mathfrak{a}_A\,,$$

such that $R \cong R_A \otimes_A K$ and, for all maximal ideals μ in a Zariski dense open subset of $\operatorname{Spec} A$, the fiber rings $R_A \otimes_A A/\mu$ are weakly F-regular rings of characteristic $p > 0$. Similarly, R is of *F-rational type* if for a dense open subset of μ, the fiber rings $R_A \otimes_A A/\mu$ are F-rational; the notions *F-pure type F-injective type* are defined similarly.

Combining results from [Ha, HW, MS, SmK] one has:

Theorem 2.5. *Let R be a ring which is finitely generated over a field of characteristic zero.*

Then R has rational singularities if and only if it is of F-rational type. If R is $\mathbb{Q}$-Gorenstein, then it has log terminal singularities if and only if it is of F-regular type.

Regarding the properties F-pure type and F-injective type, results of Hara and Watanabe [HW] and Schwede [Sc], respectively, state:

Theorem 2.6. *Let R be a ring which is finitely generated over a field of characteristic zero.*

If R is $\mathbb{Q}$-Gorenstein of F-pure type, then R has log canonical singularities. If R is of F-injective type, then it has Du Bois singularities.

Definition 2.7. Let $(R, \mathfrak{m})$ be a local or $\mathbb{N}$-graded ring. Let S_i denote the socle of $H^i_m(R)$. Then $H^i_m(R)$ is *F-unstable* if there exists $N > 0$ such that $S_i \cap F^e(S_i) \neq 0$ for each $e \geqslant N$. We say that R is F-unstable if $H^i_m(R)$ is F-unstable for all $0 \leqslant i \leqslant \dim R$.

In positive characteristic, we have the implications

$$\begin{array}{ccc} \text{F-regular} & \Rightarrow & \text{F-pure} \\ \Downarrow & & \Downarrow \\ \text{F-rational} & \Rightarrow & \text{F-injective} \end{array}$$

The work of Fedder and Watanabe [FW] shows that F-injective does not imply F-pure. However, they prove that a Cohen-Macaulay ring R is F-rational if and only if R is F-injective and F-unstable:

Definition 2.8. Following [GW1], the a_i-*invariants* of an $\mathbb{N}$-graded ring R are defined as

$$a_i(R) = \sup\{a \in \mathbb{Z} \mid \left[H^i_m(R)\right]_a \neq 0\}\,.$$

If R is an F-injective graded ring, then $a_i(R) \leqslant 0$ since the Frobenius action on $H^i_m(R)$ satisfies

$$F\colon \left[H^i_m(R)\right]_n \longrightarrow \left[H^i_m(R)\right]_{np} \qquad \text{for each } n \in \mathbb{Z}\,,$$

and $\left[H^i_m(R)\right]_{\gg 0} = 0$ since $H^i_m(R)$ is artinian.

For a graded F-injective ring R, we have that R is F-unstable if and only if $a_i(R) < 0$ for every i: If $a_i(R) = 0$, then there exists $0 \neq x \in [H^i_m(R)]_0$. Using the F-injective property, we have that $F^e(x) \neq 0$. If the degree of $r \in m$ is greater than zero, then the degree of $rF^e(x)$ is greater than zero, and hence $rF^e(x) = 0$ for every $e > 0$. We conclude that $F^e(x) \notin \operatorname{socle}\left(H^i_m(R)\right)$ and so R is not F-unstable.

If $a_i(R) < 0$, then choose homogeneous generators $x_1, \dots, x_t$ for the socle of $H^i_m(R)$. The elements x_i have negative degrees. Choose N such that

$$p^N > \max\{-\deg(x_1), \dots, -\deg(x_t)\}\,.$$

For $x \in H^i_m(R)$, we have $\deg(F^e(x)) = p^e \deg(x) \leqslant -p^e$.

In [KSSW] Kurano, Sato, Singh, and Watanabe studied properties of diagonal subalgebras of bigraded rings. Their results include:

Theorem 2.9 ([KSSW])**.** *Let K be a field, let m, n be integers with $m, n \geqslant 2$, and let*

$$\mathcal{R} = K[x_1, \dots, x_m, y_1, \dots, y_n]/(f)$$

be a normal $\mathbb{N}^2$-graded hypersurface where $\deg x_i = (1,0)$, $\deg y_j = (0,1)$, and $\deg f = (d,e) > (0,0)$. For positive integers g and h, set $\Delta = (g,h)\mathbb{Z}$. Then:

1. *The ring $\mathcal{R}_\Delta$ is Cohen-Macaulay if and only if $\lfloor (d-m)/g \rfloor < e/h$ and $\lfloor (e-n)/h \rfloor < d/g$. In particular, if $d < m$ and $e < n$, then $\mathcal{R}_\Delta$ is Cohen-Macaulay for each diagonal Δ.*

2. *The graded canonical module of $\mathcal{R}_\Delta$ is $\mathcal{R}(d-m, e-n)_\Delta$. Hence $\mathcal{R}_\Delta$ is Gorenstein if and only if $(d-m)/g = (e-n)/h$, and this is an integer.*

If K has characteristic zero, and f is a generic polynomial of degree (d, e), then:

3. *The ring $\mathcal{R}_\Delta$ has rational singularities if and only if it is Cohen-Macaulay and $d < m$ or $e < n$.*

4. *The ring $\mathcal{R}_\Delta$ is of F-regular type if and only if $d < m$ and $e < n$.*

By [KSSW], a generic hypersurface of degree $(d, e) > (0, 0)$ in m, n variables is normal precisely when

$$m > \min(2, d) \quad \text{and} \quad n > \min(2, e)$$

We shall prove the corresponding result for the F-injectivity of $\mathcal{R}_\Delta$, see Theorem 2.12.

2.2 Multigraded rings

By an $\mathbb{N}^r$-*graded ring* we mean a ring

$$\mathcal{R} = \bigoplus_{\boldsymbol{n} \in \mathbb{N}^r} \mathcal{R}_{\boldsymbol{n}}\,,$$

which is finitely generated over the subring $\mathcal{R}_{\boldsymbol{0}}$. If $(\mathcal{R}_{\boldsymbol{0}}, \mathfrak{m})$ is a local ring, then $\mathcal{R}$ has a unique homogeneous maximal ideal $\mathfrak{M} = \mathfrak{m}\mathcal{R} + \mathcal{R}_+$, where $\mathcal{R}_+ = \oplus_{\boldsymbol{n} \neq \boldsymbol{0}} \mathcal{R}_{\boldsymbol{n}}$.

For $\boldsymbol{m} = (m_1, \dots, m_r)$ and $\boldsymbol{n} = (n_1, \dots, n_r)$ in $\mathbb{Z}^r$, we say $\boldsymbol{n} > \boldsymbol{m}$ (resp. $\boldsymbol{n} \geqslant \boldsymbol{m}$) if $n_i > m_i$ (resp. $n_i \geqslant m_i$) for each i.

Let M be a $\mathbb{Z}^r$-graded $\mathcal{R}$-module. For $\boldsymbol{m} \in \mathbb{Z}^r$, we set

$$M_{\geqslant \boldsymbol{m}} = \bigoplus_{\boldsymbol{n} \geqslant \boldsymbol{m}} M_{\boldsymbol{n}}\,,$$

which is a $\mathbb{Z}^r$-graded submodule of M. One writes $M(\boldsymbol{m})$ for the $\mathbb{Z}^r$-graded $\mathcal{R}$-module with shifted grading $[M(\boldsymbol{m})]_{\boldsymbol{n}} = M_{\boldsymbol{m}+\boldsymbol{n}}$ for each $\boldsymbol{n} \in \mathbb{Z}^r$.

Let $\mathcal{R}$ be a $\mathbb{Z}^2$-graded ring and let g, h be positive integers. The subgroup $\Delta = (g, h)\mathbb{Z}$ is a *diagonal* in $\mathbb{Z}^2$, and the corresponding *diagonal subalgebra* of $\mathcal{R}$ is

$$\mathcal{R}_\Delta = \bigoplus_{k \in \mathbb{Z}} \mathcal{R}_{(gk, hk)}\,.$$

Similarly, if M is a $\mathbb{Z}^2$-graded $\mathcal{R}$-module, we set

$$M_\Delta = \bigoplus_{k \in \mathbb{Z}} M_{(gk, hk)}\,,$$

which is a $\mathbb{Z}$-graded module over the $\mathbb{Z}$-graded ring $\mathcal{R}_\Delta$.

We recall the following lemma from [KSSW]:

Lemma 2.10. *Let A and B be $\mathbb{N}$-graded normal rings, finitely generated over a field $A_0 = K = B_0$. Set $T = A \otimes_K B$. Let g and h be positive integers and set $\Delta = (g,h)\mathbb{Z}$. Let $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{m}$ denote the homogeneous maximal ideals of A, B, and T_Δ respectively. Then, for each $q \geqslant 0$ and $i, j, k \in \mathbb{Z}$, one has*

$$H^q_{\mathfrak{m}}\big(T(i,j)_\Delta\big)_k = \big(A_{i+gk} \otimes H^q_{\mathfrak{b}}(B)_{j+hk}\big) \oplus \big(H^q_{\mathfrak{a}}(A)_{i+gk} \otimes B_{j+hk}\big)$$
$$\bigoplus_{q_1+q_2=q+1} \big(H^{q_1}_{\mathfrak{a}}(A)_{i+gk} \otimes H^{q_2}_{\mathfrak{b}}(B)_{j+hk}\big)\,.$$

Proof. Let $A^{(g)}$ and $B^{(h)}$ denote the respective Veronese subrings of A and B. Set

$$A^{(g,i)} = \bigoplus_{k\in\mathbb{Z}} A_{i+gk} \qquad \text{and} \qquad B^{(h,j)} = \bigoplus_{k\in\mathbb{Z}} B_{j+hk}\,,$$

which are graded $A^{(g)}$ and $B^{(h)}$ modules respectively. Using # for the Segre product,

$$T(i,j)_\Delta = \bigoplus_{k\in\mathbb{Z}} A_{i+gk} \otimes_K B_{j+hk} = A^{(g,i)} \# B^{(h,j)}\,.$$

The ideal $A^{(g)}_+A$ is $\mathfrak{a}$-primary; likewise, $B^{(h)}_+B$ is $\mathfrak{b}$-primary. The Künneth formula for local cohomology, [GW1, Theorem 4.1.5], now gives the desired result. □

2.3 F-injectivity of diagonal subalgebras

The Cohen-Macaulay property may be characterized via local cohomology: A local or $\mathbb{N}$-graded ring $(R,\mathfrak{m})$ is Cohen-Macaulay precisely if $H^i_{\mathfrak{m}}(R) = 0$ for $i < \dim R$, and $H^{\dim R}_{\mathfrak{m}}(R) \neq 0$

The deformation property of F-injectivity is as follows: if R/xR is F-injective is R F-injective? For Cohen-Macaulay rings F-injectivity deforms; this result has been know for some time, and we provide an alternative proof below.

Theorem 2.11. *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay ring. If $x \in \mathfrak{m}$ is a nonzerodivisor and R/xR is F-injective, then R is F-injective.*

Proof. Since x is a nonzerodivisor, it does not belong to any minimal prime ideal of R. Thus, we may choose a system of parameters $\boldsymbol{x} = x_1, \ldots, x_d$ for R where $x_1 = x$. Since $\mathfrak{m}$ is the radical of the ideal $(x_1, \ldots, x_d)$, we have

$$H^i_{\mathfrak{m}}(R) = H^i_{(x_1,\ldots,x_d)}(R)\,.$$

By Lemma 1.19 the element

$$\left[\frac{r}{x_1^{m_1}\cdots x_d^{m_d}}\right]$$

of $H^d_{\mathfrak{m}}(R)$ is zero if and only if $r \in \big(x_1^{m_1}, \ldots, x_d^{m_k}\big)R$.

Suppose R is not F-injective. Then there exists a nonzero element

$$\left[\frac{y}{x_1^{m_1}\cdots x_d^{m_d}}\right] \in H^d_{\mathfrak{m}}(R)$$

that is killed by the Frobenius action $F\colon H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(R)$. Choose such an element with m_1 of least degree. Since

$$F\left[\frac{y}{x_1^{m_1}\cdots x_d^{m_d}}\right] = \left[\frac{y^p}{x_1^{pm_1}\cdots x_d^{pm_d}}\right] = 0 \qquad \text{in } H^d_{\mathfrak{m}}(R)\,,$$

it follows that

$$y^p \in \left(x_1^{pm_1},\ldots,x_d^{pm_d}\right)R\,.$$

But then

$$y^p \in \left(x_2^{pm_2},\ldots,x_d^{pm_d}\right)R/x_1R\,,$$

which implies that

$$F\left[\frac{y}{x_2^{m_2}\cdots x_d^{m_d}}\right] = \left[\frac{y^p}{x_2^{pm_2}\cdots x_d^{pm_d}}\right] = 0 \qquad \text{in } H^d_{\mathfrak{m}}(R/x_1R)\,.$$

Since R/x_1R is F-injective, we conclude

$$\left[\frac{y}{x_2^{m_2}\cdots x_d^{m_d}}\right] = 0 \qquad \text{in } H^d_{\mathfrak{m}}(R/x_1R)\,.$$

Using Lemma 1.19 and that R/x_1R is Cohen-Macaulay, we have

$$y \in \left(x_2^{m_2},\ldots,x_d^{m_d}\right)R/x_1R\,,$$

i.e., that

$$y \in \left(x_1, x_2^{m_2},\ldots,x_d^{m_d}\right)R\,.$$

But then

$$y = r_1x_1 + \sum_{i\geqslant 2} r_ix_i^{m_i}$$

so we have

$$\begin{aligned}\left[\frac{y}{x_1^{m_1}\cdots x_d^{m_d}}\right] &= \left[\frac{r_1x_1}{x_1^{m_1}\cdots x_d^{m_d}}\right] + \sum_{i\geqslant 2}\left[\frac{r_ix_i^{m_i}}{x_1^{m_1}\cdots x_d^{m_d}}\right] \\ &= \left[\frac{r_1}{x_1^{m_1-1}x_2^{m_2}\cdots x_d^{m_d}}\right]\end{aligned}$$

in $H^d_{\mathfrak{m}}(R)$, which contradicts the minimality of m_1. □

Theorem 2.12. *Let K be a field, let m, n be integers with $m, n \geqslant 2$, and let*

$$\mathcal{R} = K[x_1, \dots, x_m, y_1, \dots, y_n]/(f)$$

be a normal $\mathbb{N}^2$-graded hypersurface where $\deg x_i = (1,0)$, $\deg y_j = (0,1)$, and $\deg f = (d,e) > (0,0)$. For positive integers g and h, set $\Delta = (g,h)\mathbb{Z}$.

Set $A = K[\boldsymbol{x}]$, $B = K[\boldsymbol{y}]$, and $T = A \otimes_K B$. Let $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{m}$ denote the homogeneous maximal ideals of A, B, and T_Δ respectively.

Then $\mathcal{R}_\Delta$ is F-injective if and only if the following three maps are injective:

$$[A(-d) \otimes H^n_{\mathfrak{b}}(B(-e))]_\Delta \xrightarrow{f^{p-1}F} [A(-d) \otimes H^n_{\mathfrak{b}}(B(-e))]_\Delta \,,$$

$$[H^m_{\mathfrak{a}}(A(-d)) \otimes B(-e)]_\Delta \xrightarrow{f^{p-1}F} [H^m_{\mathfrak{a}}(A(-d)) \otimes B(-e)]_\Delta \,,$$

$$[H^m_{\mathfrak{a}}(A(-d)) \otimes H^n_{\mathfrak{b}}(B(-e))]_\Delta \xrightarrow{f^{p-1}F} [H^m_{\mathfrak{a}}(A(-d)) \otimes H^n_{\mathfrak{b}}(B(-e))]_\Delta \,.$$

Proof. We have a graded exact sequence

$$0 \longrightarrow T(-d,-e)_\Delta \xrightarrow{f} T_\Delta \longrightarrow \mathcal{R}_\Delta \longrightarrow 0\,,$$

that induces a long exact sequence on cohomology:

$$\cdots \longrightarrow H^i_{\mathfrak{m}}(T(-d,-e)_\Delta) \xrightarrow{f} H^i_{\mathfrak{m}}(T_\Delta) \longrightarrow H^i_{\mathfrak{m}}(\mathcal{R}_\Delta) \longrightarrow H^{i+1}_{\mathfrak{m}}(T(-d,-e)_\Delta) \xrightarrow{f} \cdots\,.$$

By Lemma 2.10 we have,

$$H^q_{\mathfrak{m}}\big(T(i,j)_\Delta\big)_k = \big(A_{i+gk} \otimes H^q_{\mathfrak{b}}(B)_{j+hk}\big) \oplus \big(H^q_{\mathfrak{a}}(A)_{i+gk} \otimes B_{j+hk}\big)$$
$$\bigoplus_{q_1+q_2=q+1} \big(H^{q_1}_{\mathfrak{a}}(A)_{i+gk} \otimes H^{q_2}_{\mathfrak{b}}(B)_{j+hk}\big)\,.$$

Examining this decomposition, we see that

$$H^q_{\mathfrak{m}}\big(T_\Delta\big) = \begin{cases} 0 & \text{if } q \neq m+n-1\,, \\ H^m_{\mathfrak{a}}(A)_{i+gk} \otimes H^n_{\mathfrak{b}}(B)_{j+hk} & \text{if } q = m+n-1\,, \end{cases}$$

in particular, T_Δ is a Cohen-Macaulay ring of dimension $m+n-1$. It follows that

$$H^i_{\mathfrak{m}}(\mathcal{R}_\Delta) \cong H^{i+1}_{\mathfrak{m}}(T(-d,-e)_\Delta) \qquad \text{for } i \leqslant m+n-3\,,$$

and that $H^{m+n-2}_{\mathfrak{m}}(\mathcal{R}_\Delta)$ is the kernel of

$$H^{m+n-1}_{\mathfrak{m}}(T(-d,-e)_\Delta) \xrightarrow{f} H^{m+n-1}_{\mathfrak{m}}(T_\Delta)\,.$$

To determine the Frobenius action on the modules $H^i_{\mathfrak{m}}(\mathcal{R}_\Delta)$, note that we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} \longrightarrow & H^i_{\mathfrak{m}}(\mathcal{R}_\Delta) & \longrightarrow & H^{i+1}_{\mathfrak{m}}(T(-d,-e)_\Delta) & \xrightarrow{f} & H^{i+1}_{\mathfrak{m}}(T_\Delta) & \longrightarrow \\ & \Big\downarrow F & & \Big\downarrow f^{p-1}F & & \Big\downarrow F & \\ \longrightarrow & H^i_{\mathfrak{m}}(\mathcal{R}_\Delta) & \longrightarrow & H^{i+1}_{\mathfrak{m}}(T(-d,-e)_\Delta) & \xrightarrow{f} & H^{i+1}_{\mathfrak{m}}(T_\Delta) & \longrightarrow \end{array} \tag{2.12.1}$$

For $i \leqslant m+n-3$, it follows that $F\colon H^i_{\mathfrak{m}}(\mathcal{R}_\Delta) \longrightarrow H^i_{\mathfrak{m}}(\mathcal{R}_\Delta)$ is injective precisely if

$$f^{p-1}F\colon H^{i+1}_{\mathfrak{m}}(T(-d,-e)_\Delta) \longrightarrow H^{i+1}_{\mathfrak{m}}(T(-d,-e)_\Delta)$$

is injective. By Lemma 2.10, the modules of relevance are $H^{i+1}_{\mathfrak{m}}(T(-d,-e)_\Delta)$ for $i \leqslant m+n-3$ are

$$H^m_{\mathfrak{m}}(T(-d,-e)_\Delta) = [H^m_{\mathfrak{a}}(A(-d)) \otimes B(-e)]_\Delta$$

and

$$H^n_{\mathfrak{m}}(T(-d,-e)_\Delta) = [A(-d) \otimes H^n_{\mathfrak{b}}(B(-e))]_\Delta\,.$$

Moreover, Lemma 2.10 gives us the identifications

$$H^{m+n-1}_{\mathfrak{m}}(T(-d,-e))_\Delta = [H^m_{\mathfrak{a}}(A(-d)) \otimes H^n_{\mathfrak{b}}(B(-e))]_\Delta$$

and

$$H^{m+n-1}_{\mathfrak{m}}(T)_\Delta = [H^m_{\mathfrak{a}}(A) \otimes H^n_{\mathfrak{b}}(B)]_\Delta$$

Thus, for index $i = m+n-2$ the commutative diagram (2.12.1) gives us

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^{m+n-2}_{\mathfrak{m}}(\mathcal{R}_\Delta) & \longrightarrow & [H^m_{\mathfrak{a}}(A(-d)) \otimes H^n_{\mathfrak{b}}(B(-e))]_\Delta & \xrightarrow{f} & [H^m_{\mathfrak{a}}(A) \otimes H^n_{\mathfrak{b}}(B)]_\Delta \\ & & \Big\downarrow F & & \Big\downarrow f^{p-1}F & & \Big\downarrow F \\ 0 & \longrightarrow & H^{m+n-2}_{\mathfrak{m}}(\mathcal{R}_\Delta) & \longrightarrow & [H^m_{\mathfrak{a}}(A(-d)) \otimes H^n_{\mathfrak{b}}(B(-e))]_\Delta & \xrightarrow{f} & [H^m_{\mathfrak{a}}(A) \otimes H^n_{\mathfrak{b}}(B)]_\Delta \end{array}$$

The map on the right is injective; thus

$$F\colon H^{m+n-2}_{\mathfrak{m}}(\mathcal{R}_\Delta) \longrightarrow H^{m+n-2}_{\mathfrak{m}}(\mathcal{R}_\Delta)$$

is injective precisely if

$$f^{p-1}F\colon [H^m_{\mathfrak{a}}(A(-d)) \otimes H^n_{\mathfrak{b}}(B(-e))]_\Delta \longrightarrow [H^m_{\mathfrak{a}}(A(-d)) \otimes H^n_{\mathfrak{b}}(B(-e))]_\Delta$$

is injective. $\square$

CHAPTER 3

RINGS OF INVARIANTS AND TIGHT CLOSURE THEORY

In this chapter we use tight closure theory to study properties of rings of invariants. We compare the properties of R^G and R^H, where H is a p-Sylow subgroup of G.

3.1 Introduction

Invariant theory has a rich history, for example, Hilbert's 14-th problem asks: If G is an algebraic group acting on a polynomial ring R, is the ring of invariants R^G finitely generated? Nagata produced an example in which the ring of invariants is not finitely generated. However, when G is a finite group, Noether proved that R^G is indeed finitely generated.

Let G be a subgroup of $\mathrm{GL}_n(K)$. Then G has a natural action on a polynomial ring $R = K[x_1, \ldots, x_n]$. The ring R^G is the ring of invariants, i.e.,

$$R^G = \{r \in R \mid g(r) = r \text{ for all } g \in G\}\,.$$

Example 3.1. Let $R = \mathbb{R}[x_1, x_2, y_1, y_2]$ and $G = \mathbb{R} \smallsetminus \{0\}$ with the action is given by:

$$\lambda(x_i) = \lambda x_i \quad \text{and} \quad \lambda(y_i) = \lambda^{-1} y_i \text{ for } i = 1, 2\,.$$

We may compute R^G by considering the action of G on monomials:

$$\lambda(x_1^i x_2^j y_1^k y_2^l) = \lambda^{i+j-k-l} x_1^i x_2^j y_1^k y_2^l\,.$$

For the monomial above to be fixed by the action of G, we require

$$\lambda^{i+j-k-l} = 1$$

which occurs if $i + j = k + l$. It follows that

$$R^G = \mathbb{R}[x_1 y_1, x_2 y_1, x_1 y_2, x_2 y_2]\,.$$

One can show that R^G has dimension 3 and is a hypersurface.

When R^G is a direct summand of R, the ring of invariants R^G inherits several properties of R. In particular, if R is noetherian then R^G is noetherian as well.

Definition 3.2. Let $R \subseteq S$ be rings. We say S is a *direct summand* of R if there exists an S-linear map $\rho\colon R \longrightarrow S$ such that $\rho(s) = s$ for all $s \in S$.

Suppose S is a direct summand of R, and I is an ideal of S. Then

$$IR \cap S = I.$$

Let a group G act on a noetherian ring R, and suppose R^G is a direct summand of R. Let

$$I_1 \subset I_2 \subset I_3 \subset \cdots$$

be a chain of ideals of R^G. We expand these ideals to ideals of R

$$I_1R \subset I_2R \subset I_3R \subset \cdots .$$

The chain of ideals of R terminates since R is noetherian. The property that $I_jR \cap R^G = I_j$ implies that the original chain terminates. Hence, R^G is noetherian.

If R is a polynomial ring in four variables over $\mathbb{F}_2$, and G is the cyclic group of order 4 acting by permuting the four variables, then Bertin [Be] showed that the ring of invariants R^G is a unique factorization domain which is not Cohen-Macaulay; a key point here is that R^G is not a direct summand of R.

In the next example, we consider the action of an infinite group G on a Cohen-Macaulay ring R. The ring of invariants here is not Cohen-Macaulay, even though R^G is a direct summand of R.

Example 3.3. Let $R = \mathbb{R}[x_1, x_2, x_3, y_1, y_2]/(x_1^3 + x_2^3 + x_3^3)$ and $G = \mathbb{R} \smallsetminus \{0\}$. The action is given by:

$$\lambda(x_i) = \lambda x_i \;\text{ and }\; \lambda(y_j) = \lambda^{-1} y_j \;\text{ for }\; i = 1, 2, 3,\; j = 1, 2.$$

We compute R^G by considering the action of G on monomials.

$$\lambda(x_1^i x_2^j x_3^r y_1^k y_2^l) = \lambda^{i+j-k-l} x_1^i x_2^j y_1^k y_2^l.$$

For the monomial to be fixed by the action of G we require

$$\lambda^{i+j+r-k-l} = 1$$

which occurs when $i + j + r = k + l$. It follows that

$$R^G = \mathbb{R}[x_1y_1, x_2y_1, x_3y_1, x_1y_2, x_2y_2, x_3y_2].$$

The ring R is a hypersurface and is therefore Cohen-Macaulay. One can see that R^G is a direct sum of R, however R^G is not Cohen-Macaulay. To see this note that x_1y_1, x_2y_2, $x_1y_2 + x_2y_1$ is a system of parameters for R^G. These parameters have the relation

$$x_3y_1x_3y_2(x_1y_2 + x_2y_1) = (x_3y_2)^2x_1y_1 + (x_3y_1)^2x_2y_2.$$

This shows that the system of parameters do not form a regular sequence on R^G. Hence R^G need not be Cohen-Macaulay even when R^G is a direct summand of a Cohen-Macaulay ring R.

3.2 Preliminaries

Let R be a polynomial ring over a field of characteristic $p > 0$. If the order of the group is relatively prime to the characteristic p of the field, then there is an R^G-linear retraction $\rho\colon R \longrightarrow R^G$, the *Reynolds operator*. This retraction makes R^G a direct summand of R as an R^G-module, and so R^G is F-regular.

Let $R = \mathbb{F}_q[x_1, \ldots, x_n]$ be a polynomial ring in n variables over a finite field of characteristic $p > 0$. Let G be a subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$ and H a subgroup of G. The *transfer map* is defined as $TR\colon R^H \longrightarrow R^G$, where

$$TR(f) = \sum_{gH \in G/H} g(f)\,.$$

Here g runs over all left coset representatives of H in G. The map TR is independent of the choice of coset representatives. Note that TR is an R^G-module homomorphism. The composition

$$R^G \hookrightarrow R^H \xrightarrow{TR} R^G.$$

is multiplication by the index of H in G, denoted by $|G : H|$. If this index is invertible in $\mathbb{F}_q$, then the transfer map is surjective, and the map

$$\rho = \frac{1}{|G : H|}TR$$

is an idempotent projection whose image is R^G; in this case R^G is a direct summand of R^H; see [SmL].

When $H = \{0\}$ and $|G|$ is invertible, the map $\rho\colon R \longrightarrow R^G$ above is precisely the Reynolds operator.

Theorem 3.4. *Let G be a finite group acting on a ring R of characteristic p, and H a subgroup. If $|G:H|$ is invertible in $\mathbb{F}$, then R^G is a direct summand of R^H, the splitting being provided by*

$$\rho = \frac{1}{|G:H|} TR\,.$$

This does not provide any insight as to whether R^G is a direct summand of R when the characteristic of R divides the order of G. This question can be answered using tight closure theory. More specifically, the following theorem follows from [HH2] and [HH4].

Theorem 3.5. *Let K be a field of positive characteristic, and G be a finite group acting on a polynomial ring $R = K[x_1, \ldots, x_n]$ by degree preserving K-algebra automorphisms. Then R^G is a direct summand of R if and only if R^G is weakly F-regular.*

We will use the map ρ to study when rings of invariants are F-rational, F-regular, and F-pure. These properties were studied in some detail by Glassbrenner and Singh. Glassbrenner showed that rings of invariants of the alternating group A_n acting on a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field of characteristic p give examples of F-pure rings that are not F-regular, [Gl]. Singh considered the action of the symplectic group on R, and proved that the ring of invariants in this case is not F-pure. He further studied the action of the alternating group; the F-regularity here depends on the choice of p and on the size of the alternating group.

Example 3.6. Let $R = K[x_1, \ldots, x_n]$, and the characteristic of K be other than 2. For $n \geqslant 3$ let the alternating group act on R by permuting the variables. Let

$$\triangle = \prod_{i<j}(x_i - x_j) \in R$$

Let e_i be the elementary symmetric functions. The fixed group of the action of S_n on R is the polynomial ring $R^G = K[e_1, \ldots, e_n]$. One can see that every even cycle of S_n fixes $\triangle$. The ring of invariants of the alternating group A_n is $R^{A_n} = K[e_1, \ldots, e_n, \triangle]$. One can show that R^{A_n} is a hypersurface. To see this note that for each $g \in S_n$, one has $g(\triangle) = \operatorname{sgn}(g)\triangle$, and therefore, $\triangle^2$ is fixed by every element of S_n. This computation shows that R^{A_n} is a hypersurface with defining equation $\triangle^2 - f(e_1, \ldots, e_n) = 0$.

Theorem 3.7 ([Si1])**.** *$G = Sp_{2n}(\mathbb{F}_q)$ be the symplectic group with its natural action on the polynomial ring $R = \mathbb{F}_q[x_1, \ldots, x_{2n}]$. If $n > 2$ and $q \geqslant 4n - 4$, then the ring of invariants R^G is not F-pure.*

Theorem 3.8 ([Si1]). *Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring in n variables over a field of characteristic p, an odd prime, and let the alternating group A_n, act on R by permuting the variables. Then the invariant subring R^{A_n} is F-regular if and only if the order of the group $|A_n|$ is relatively prime to p.*

3.3 Invariants of p-Sylow subgroups

We consider the ring R^G and want to examine whether this ring is F-regular, F-rational, or F-pure. We show that it is sufficient to study the ring R^H and show that it has the desired property.

Theorem 3.9. *Let $G < \mathrm{GL}_n(K)$ act on a polynomial ring $R = K[x_1, \ldots, x_n]$, where K is a field of characteristic p. Let H be a p-Sylow subgroup of G. Then:*

1. *If R^H is F-regular, so is R^G.*

2. *If R^H is F-pure, so is R^G.*

3. *If R^H is F-rational, so is R^G.*

Proof. Let the order of G be mp^a, where p does not divide m. Consider the transfer map

$$TR_H^G(f) = \sum_{gH \in G/H} g(f).$$

Since $|G : H| = m$ is invertible in K, the map

$$\frac{1}{m} TR.$$

shows that R^G is a direct summand of R^H.

A direct summand of an F-regular ring is F-regular, hence, (1) follows. A direct summand of an F-pure ring is F-pure, as may be found in [HR], giving us (2). Since R^H and R^G have the same dimension, a system of parameters for R^G is also a system of parameters for R^H. Hence, if R^H is F-rational, so is R^G. □

Theorem 3.10. *If T and Q are two p-Sylow subgroups then the rings R^T and R^Q are isomorphic. In particular, if R^T is F-rational, F-regular, or F-pure then so is R^Q.*

Proof. This follows from the fact that p-Sylow subgroups are conjugate. □

We conclude with an example where R^G is F-regular, but R^H is not F-regular for any p-Sylow subgroup H.

Example 3.11. Let $R = \mathbb{F}_3[x_1, x_2, x_3]$. Let $G = S_3$, the symmetric group on 3 letters, and consider A_3 the 3-Sylow subgroup of S_3. Then

$$R^{S_3} = \mathbb{F}_3[e_1, e_2, e_3]\,,$$

and

$$R^{A_3} = F_3[e_1, e_2, e_3, \triangle]$$

where e_i are the elementary symmetric functions on three variables, and

$$\triangle = (x - y)(x - z)(y - z)\,.$$

Then R^{S_3} is F-regular in fact it is a polynomial ring, whereas R^{A_3} is not F-regular by [Si1].

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